A parametric level-set method for partially discrete tomography

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Objective

- Partially discrete objects exist in many applications, including, **image processing**, **medical imaging** and material science.
- In many of these applications, tomographic projection data is **limited**, either due to limited-view or sparse sampling.
- The current *state-of-the-art* algorithms fails to reconstruct these **objects** from limited data.
- We develop new algorithm to capture the **shape** of such objects with little *prior* information.

Introduction

The need to reconstruct (quantitative) images of an object from tomographic measurements appears in many applications. At the heart of many is a projection model based on the **Radon transform**:

$$p_i = \int_{\mathcal{D}} u(\mathbf{x})\delta(s_i - \mathbf{n}(\theta_i)\cdot\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

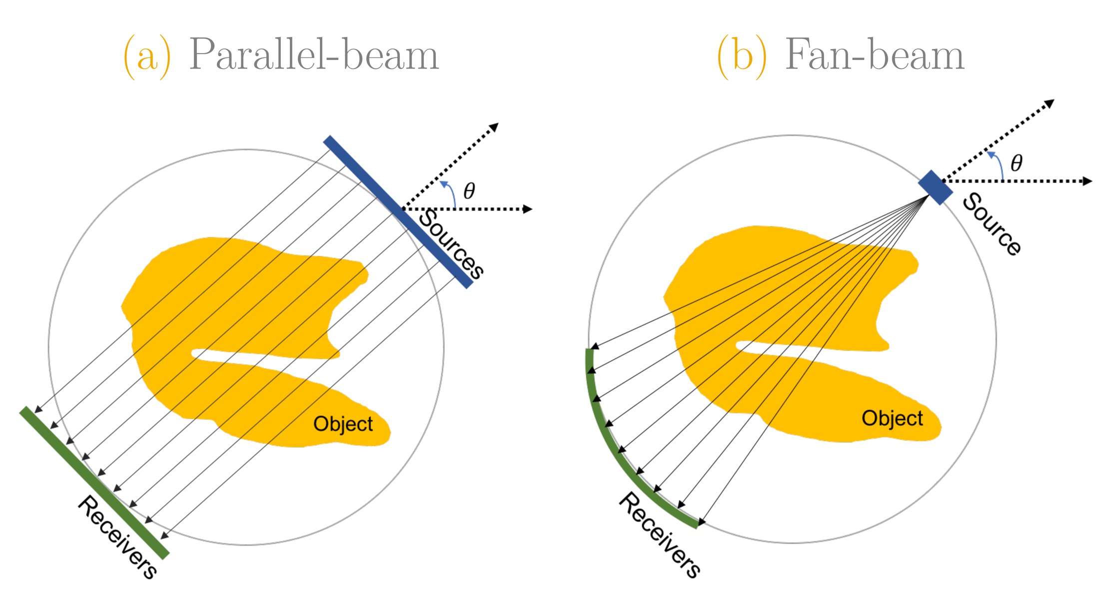

Figure 1: Tomography acquisition systems

An *Algebraic reconstruction* technique is used for dealing with non-regularly sampled or missing data,

$$u(\mathbf{x}) = \sum_{j=1}^{n} u_j b(\mathbf{x} - \mathbf{x}_j), \quad \Longrightarrow \quad \mathbf{p} = W\mathbf{u}$$

The system of equations is typically inconsistent and underdetermined. A standard approach is to formulate **regularized least-squares problem**:

$$\min_{\mathbf{u}} \underbrace{\frac{1}{2}\|W\mathbf{u} - \mathbf{p}\|_2^2}_{\text{Data-Misfit}} + \underbrace{\lambda\|R\mathbf{u}\|}_{\text{Regularization}}$$

Partially discrete objects consists of a region of constant density embedded in a continuously varying background. We therefore propose the following parametrization:

$$u(\mathbf{x}) = \begin{cases} u_1 & \text{if } \mathbf{x} \in \Omega, \\ u_0(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Constant density region is characterized using **level-set function**, $\Omega = \{\mathbf{x} \,|\, \phi(\mathbf{x}) \geq 0\}$.

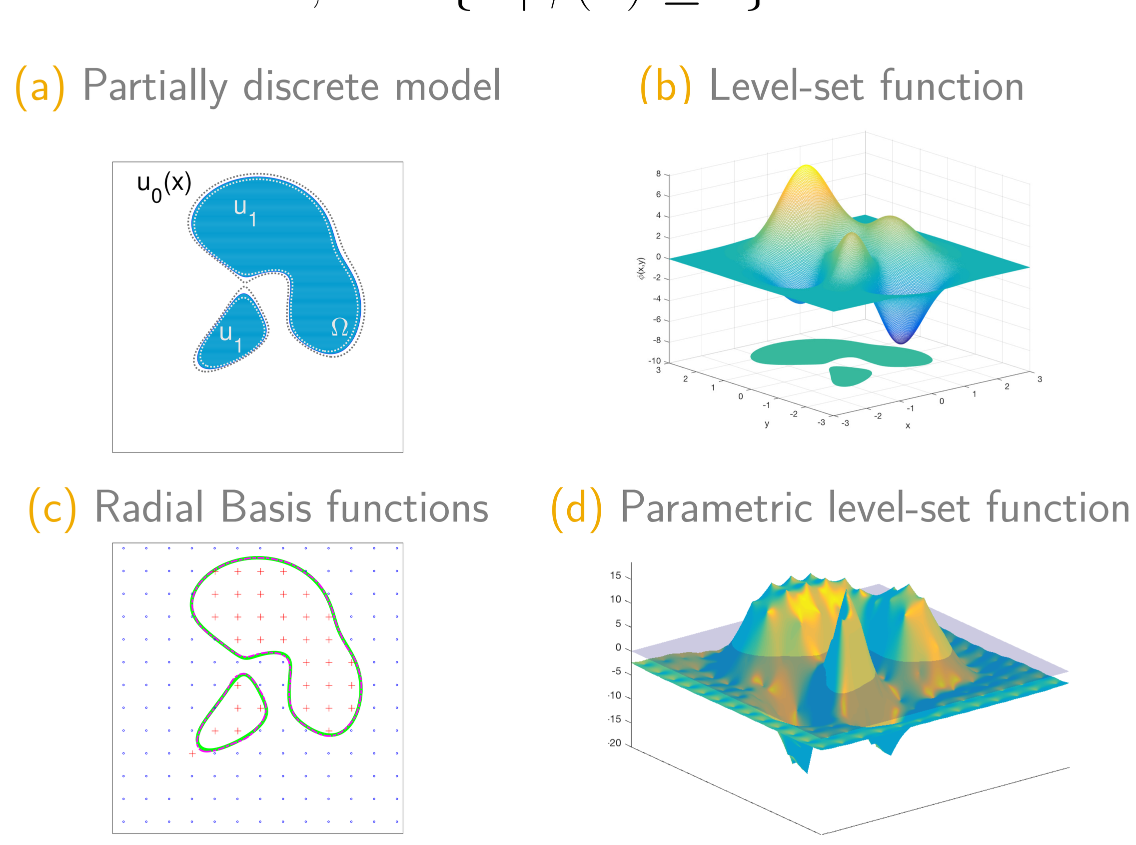

Figure 2: Example of Parametric Level-Set Method

Results

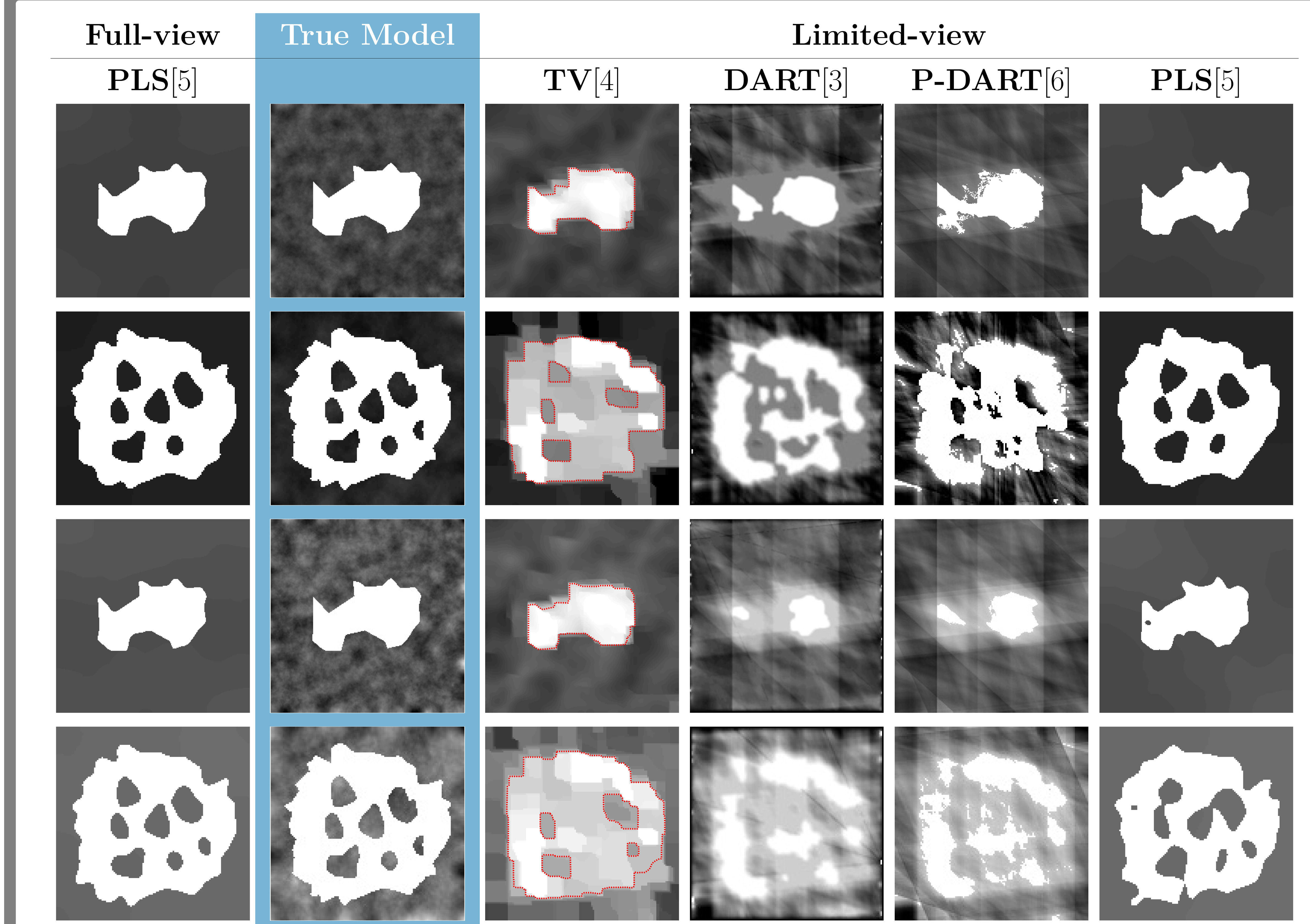

Figure 3: Reconstructions with noisy (SNR=10 dB) full-view ($\theta \in [0, \pi]$) and limited-view ($\theta \in [0, \frac{2\pi}{3}]$) data. All the models have a resolution of 256×256 pixels. Full-view: 180 equidistant projections, Limited-view: 5 equidistant projections.

Parametric Level-Set Method

In terms of the level-set function, we can express u as

$$\begin{aligned} u(\mathbf{x}) &= (\mathbf{1} - h_\epsilon(\phi(\mathbf{x})))u_0(\mathbf{x}) + h_\epsilon(\phi(\mathbf{x}))u_1, \\ &= B(\mathbf{u}_0, u_1, \phi). \end{aligned}$$

The level-set function is parametrized using radial basis functions (RBF) [1]:

$$\phi(\mathbf{x}) = \sum_{j=1}^{n'} \alpha_j \Psi(\beta_j \|\mathbf{x} - \boldsymbol{\chi}_j\|_2).$$

The shape reconstruction problem can now be cast in terms of **RBF coefficients** α, given $\mathbf{u}_0$ and u_1.

Approximation to Heaviside function

Lemma: The width of the level-set boundary layer, Δ, satisfies

$$\Delta \geq \epsilon / \|\nabla\phi\|_\infty. \quad \square$$

We approximate this using upper and lower bounds:

$$\epsilon = \kappa(\max(\phi) - \min(\phi))/\Delta x.$$

Joint Reconstruction Algorithm

Reconstructing both the shape and the background parameter can be cast as a **bi-level optimization** problem

$$\min_{\mathbf{u}_0, \boldsymbol{\alpha}} \left\{ f(\boldsymbol{\alpha}, \mathbf{u}_0) := \tfrac{1}{2}\|WB(\mathbf{u}_0, u_1, \boldsymbol{\alpha}) - \mathbf{p}\|_2^2 + \tfrac{\lambda}{2}\|L\mathbf{u}_0\|_2^2 \right\}$$

Exploiting the fact that the problem has a closed-form solution in $\mathbf{u}_0$ for each $\boldsymbol{\alpha}$, we introduce a reduced objective:

$$\min_{\boldsymbol{\alpha}} \left\{ \overline{f}(\boldsymbol{\alpha}) = \min_{\mathbf{u}_0} f(\boldsymbol{\alpha}, \mathbf{u}_0) \right\}.$$

We use modified *Gauss-Newton* algorithm[2] to find a minimizer of $\overline{f}$.

Conclusion

- Partially discrete objects modeled as a constant-valued shape embedded in a background.
- The shape is represented using a **level-set** of handful compact radial basis functions.
- The reconstruction problem can be efficiently solved using a **variable projection** approach.
- The proposed approach can **outperform** other popular methods for (partially) discrete tomography.
- Future research is directed at making the method more efficient.

Acknowledgements

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