Convex Formulation For Discrete Tomography

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Discrete Tomography

Discrete tomography deals with reconstructing objects with constant greyvalues from their projection data [6]. Assuming Gaussian noise in the data ($\mathbf{p}$), the resulting inverse problem is:

$$\min_{\mathbf{u}\in\{\rho_1,\rho_2,\ldots,\rho_n\}^n} \{\mathcal{F}(\mathbf{u}) = \|W\mathbf{u}-\mathbf{p}\|^2\} \quad \text{(discrete)}$$

$W \in \mathbb{R}^{m\times n}$ is a projection operator.

Connection to level-set method:
Level-set method was introduced for modeling time-varying objects [5]. An object reconstruction problem with constant density can be modeled using level-set method as follows:

$$\min_{\mathbf{u},\boldsymbol{\phi}} \|W\mathbf{u}-\mathbf{p}\|_2^2$$
$$\text{subject to } \mathbf{u} = h_s(\boldsymbol{\phi})$$

Here, $h_s(\cdot)$ represents signum function. This introduces non-linearity in the inverse problem.

Challenges:

- Discrete tomography: A combinatorial problem
- Level-set problem: Non-linear inverse problem
- Problem is usually highly under-determined! Hence, many solutions may exist.

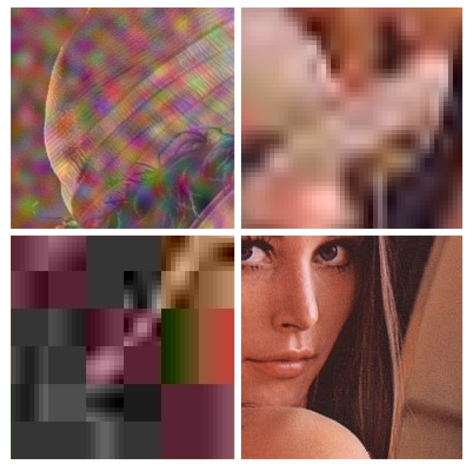
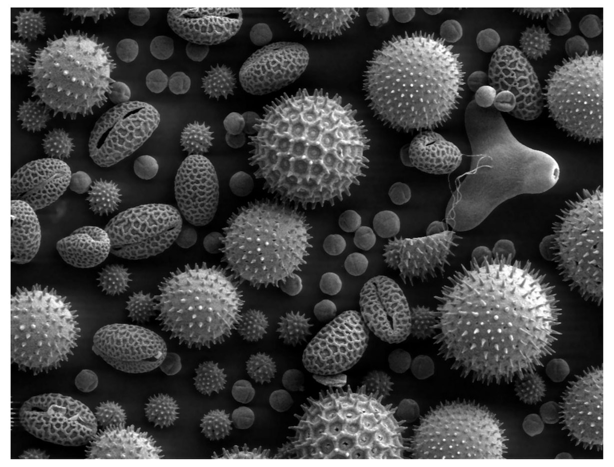
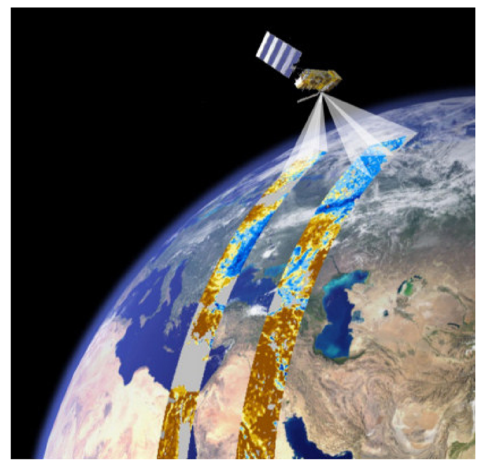


Image Processing



Electron microscopy

Computerized Tomography



Remote Sensing

Dual formulation

Primal Problem

$$\min_{\mathbf{u},\boldsymbol{\phi}} \quad \tfrac{1}{2}\|W\mathbf{u}-\mathbf{p}\|^2$$
$$\text{such that} \quad \mathbf{u} = h_s(\boldsymbol{\phi})$$

Lagrangian function:

$$\mathcal{L}(\mathbf{u},\boldsymbol{\phi},\boldsymbol{\lambda}) = \tfrac{1}{2}\|W\mathbf{u}-\mathbf{p}\|^2 + \boldsymbol{\lambda}^T(\mathbf{u}-h_s(\boldsymbol{\phi}))$$

Dual Problem

$$\min_{\boldsymbol{\mu}} \quad \tfrac{1}{2}\|\boldsymbol{\mu}-\mathbf{p}\|^2 + \|W^T\boldsymbol{\mu}\|_1$$

Primal-Dual relation: $\mathbf{u}^* = h_s(W^T\boldsymbol{\mu}^*)$

Generalized LASSO

The generalized LASSO (aka gLASSO) is a generalized version of least absolute shrinkage and selection operator (LASSO) where sparsity is penalized on a projected space of variable.

$$\min_{\mathbf{u}\in\mathbb{R}^n} \{\|A\mathbf{u}-\mathbf{b}\|_2^2 + \lambda\|B\mathbf{u}\|_1\}$$

$A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times n}$ are general matrices. The sparsity (ℓ_1-norm) acts on range (image) of B.

Numerical Methods:

- solution path [8]
- ADMM / Split Bregman [2, 4]
- Smooth Approximation of ℓ_1 [7]

Conclusions and Outlook

- The discrete tomography problem is inherently a **combinatorial** problem, while level-set method is a **non-linear** inverse problem.
- Recently, **heuristic**-based algorithm has been proposed to solve this combinatorial problem [1].
- We derive a dual formulation for such a problem (with **strong** duality), which is **convex**!

Future Work:

- study the *behavior* of discrete tomography problem: convergence, uniqueness, etc
- case study: W is a *rank-deficient* matrix.
- Formulating a convex problem for discrete tomography with *multiple* greyvalues.

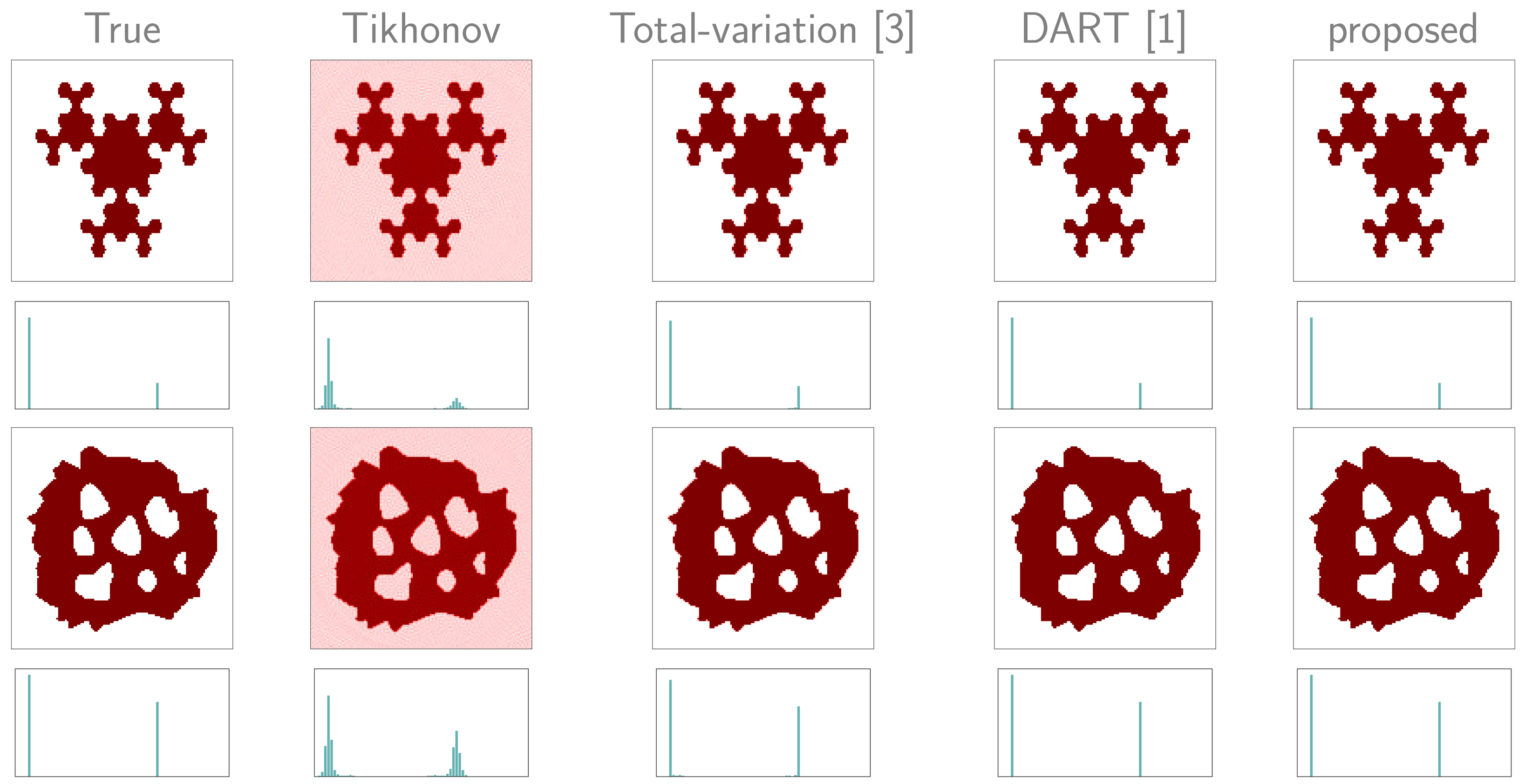
Acknowledgment

Project funded by NWO/FOM in joint collaboration with Shell Global Solutions (14CSER028). 2^{nd} author supported by NWO Grant 613.009.032. We would like to thank Joost Batenburg (CWI) and Nick Luiken (UU) for helpful discussions.

Results

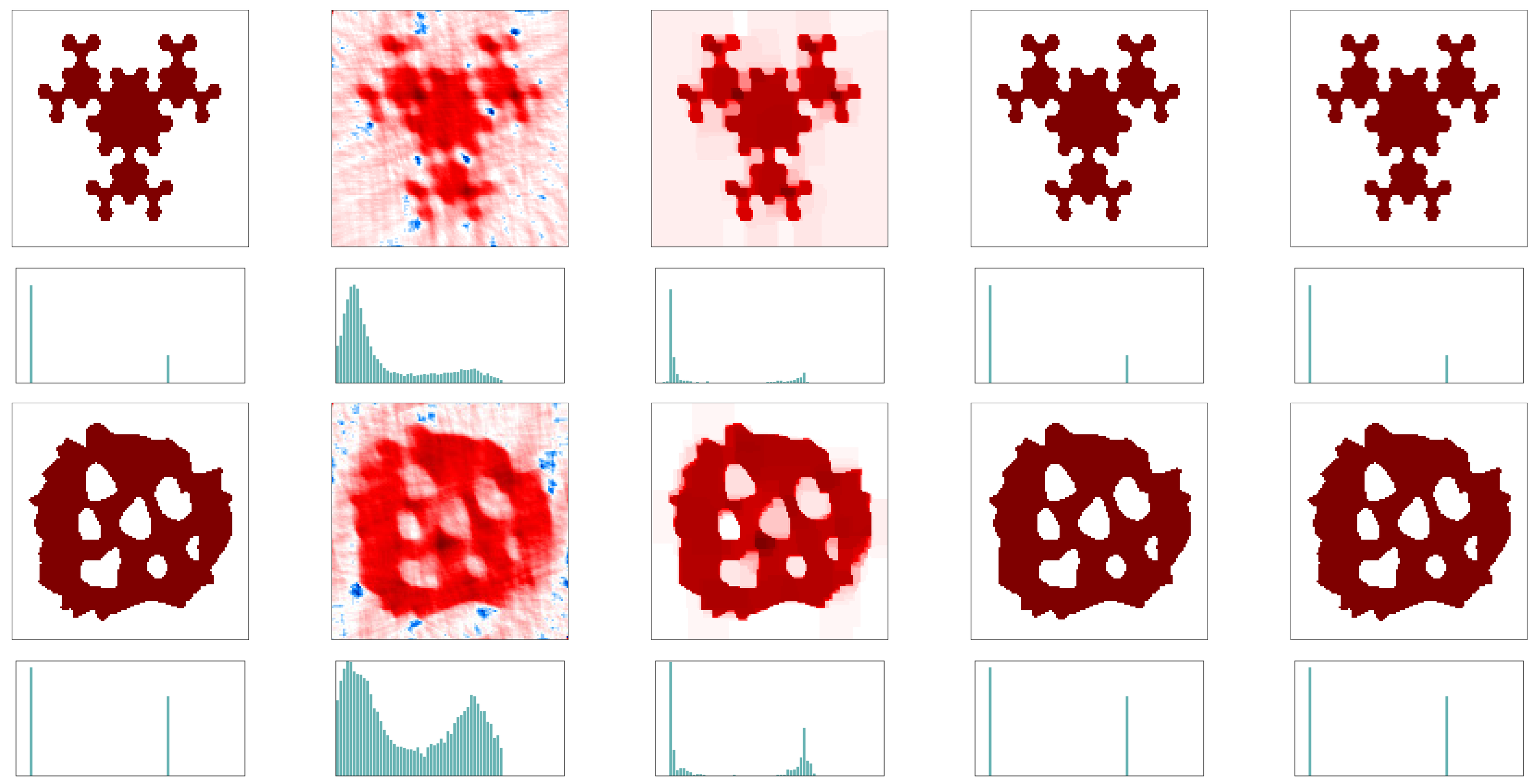
Full-view Tomography:

Highly sampled projection data from 0 to π with a Gaussian noise of 50 dB SNR. The models have a resolution of 128×128 pixels. W has a size of $(128 \times 180) \times (128 \times 128)$ making it a over-determined system.



Limited-data Tomography:

Sparsely sampled projection data from 0 to $\frac{2\pi}{3}$ with a Gaussian noise of 50 dB SNR. 20 angles are considered. W has a size of $(128 \times 20) \times (128 \times 128)$ - an under-determined system.



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